$$\Lambda = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

$$l = |\vec{L}|$$

$$\frac{d}{dt}\left(\frac{\partial \Lambda}{\partial \dot{q}_i}\right) - \frac{\partial \Lambda}{\partial q_i} = \phi$$

$$V(r) = \gamma \frac{M}{r}$$

$$\frac{\partial \Lambda}{\partial \dot{\theta}} = + \frac{1}{2} m \cdot 2 r^2 \dot{\theta} = + m r^2 \dot{\theta} \Rightarrow \frac{d}{dt}(m r^2 \dot{\theta}) = \phi$$

$$\boxed{m r^2 \dot{\theta} = l = const}$$

$$\frac{\partial \Lambda}{\partial \dot{r}} = m\dot{r} \qquad \frac{\partial \Lambda}{\partial r} = + m r \dot{\theta}^2 - \frac{\partial V}{\partial r}$$

$$\frac{d}{dt}(m\dot{r}) - m r \dot{\theta}^2 + \frac{\partial V}{\partial r} = \phi$$

$$m\ddot{r} - m r \dot{\theta}^2 + \frac{\partial V}{\partial r} = \phi$$

$$f(r) = -\frac{\partial V}{\partial r}$$

$$m\ddot{r} - m r \dot{\theta}^2 = f(r)$$

$$\boxed{m\ddot{r} - \frac{l^2}{m r^3} = f(r)} \Rightarrow m\ddot{r} + \frac{d}{dr}\left[V(r) + \frac{1}{2}\frac{l^2}{m r^2}\right] = \phi \quad \Big/ \frac{dr}{dt}$$

$$E = \frac{1}{2} m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

$$m\ddot{r}\dot{r} = -\frac{dr}{dt}\frac{d}{dr}\left[V(r) + \frac{1}{2}\frac{l^2}{m r^2}\right]$$

$$m\ddot{r}\dot{r} = \frac{d}{dt}\left(\frac{1}{2} m\dot{r}^2\right) \Longleftarrow m\ddot{r}\dot{r} = -\frac{d}{dt}\left[V(r) + \frac{1}{2}\frac{l^2}{m r^2}\right]$$

$$\frac{d}{dt}\left(\frac{1}{2} m\dot{r}^2\right) = -\frac{d}{dt}\left[V(r) + \frac{1}{2}\frac{l^2}{m r^2}\right]$$

$$\underbrace{\frac{1}{2} m\dot{r}^2 + \frac{1}{2}\frac{l^2}{m r^2}}_{T} + V(r) = const$$

$$T + V(r) = E$$

$$\dot{r}^2 = \frac{2}{m}\left[E - V(r) - \frac{1}{2} m r^2 \dot{\theta}^2\right] = \frac{2}{m}\left[E - V(r) - \frac{l^2}{\ldots}\right]$$

$$\dot{r} = \sqrt{\frac{2}{m}\left[E - V(r) - \frac{l^2}{2mr^2}\right]}$$

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left[E - V(r) - \frac{l^2}{2mr^2}\right]}} \Rightarrow t = \int \frac{dr}{\sqrt{\frac{2}{m}\left(E - V - \frac{l^2}{2mr^2}\right)}}$$

$$\boxed{E = -|E|} \qquad \boxed{V = -\frac{M}{r}}$$

$$t = \sqrt{\frac{m}{2|E|}} \int \frac{dr}{\sqrt{-1 + \frac{M}{r|E|} - \frac{l^2}{2mr^2|E|}}} = \sqrt{\frac{m}{2|E|}} \int \frac{r\,dr}{\sqrt{\underbrace{-r^2 + \frac{M}{|E|}r - \frac{l^2}{2m|E|}}}}$$

$$\underbrace{-r^2 + 2\frac{M}{2r|E|} - \frac{M^2}{4|E|^2}}_{+ \quad - \quad +} + \frac{M^2}{4|E|^2} - \frac{l^2}{2m|E|} =$$

$$= -\left(r - \frac{M}{2|E|}\right)^2 + \frac{M^2}{4|E|^2} - \frac{l^2}{\underbrace{2m|E|}_{-2mE}} = \quad , \quad \boxed{a = \frac{M}{2|E|}}$$

$$= -(r-a)^2 + \frac{M^2}{4E^2}\left(1 + 2\frac{El^2}{M^2 m}\right) = \quad , \quad \boxed{e = \sqrt{1 + 2\frac{El^2}{mM^2}}}$$

$$= -(r-a)^2 + a^2e^2$$

$$\boxed{t = \sqrt{\frac{m}{2|E|}} \int \frac{r\,dr}{\sqrt{a^2e^2 - (r-a)^2}}}$$

$$r - a = -ae\cos u$$

$$\boxed{r = a(1 - e\cos u)} \Rightarrow dr = ae\sin u\,du$$

$$t = \sqrt{\frac{m}{2|E|}} \int \frac{a(1 - e\cos u)}{\sqrt{a^2e^2 - a^2e^2\cos^2 u}} \cdot ae\sin u\,du = \sqrt{\frac{m}{2|E|}} \int \frac{a(1 - e\cos u)}{\cancel{ae}\sqrt{\cancel{1 - \cos^2 u}}}\,\cancel{ae}\,\cancel{\sin u}\,du$$

$$t = a\sqrt{\frac{m}{2|E|}} \int (1 - e\cos u)\,du = a\sqrt{\frac{m}{2|E|}}\,(u - e\sin u)$$

$$t = \frac{M}{2|E|}\sqrt{\frac{m}{2|E|}}\,(u - e\sin u)$$

$$t = \frac{M\sqrt{m}}{(-2E)^{3/2}}\,(u - e\sin u) = C\,(u - e\sin u)$$

3a

$$u - \xi = e\sin u$$

$$\frac{t}{C} = u - e\sin u \Rightarrow \boxed{\xi = u - e\sin u} \qquad u = u(\xi) \;!$$

$$e\sin u = \sum_{k=1}^{\infty} 2 b_k \sin(k\xi) \qquad \begin{cases} \xi = \phi \Rightarrow e\sin u = \phi \\ \xi = \pi \Rightarrow e\sin u = \phi \end{cases}$$

$$b_k = \frac{1}{\pi}\int_0^{\pi} e\sin u \cdot \sin(k\xi)\,d\xi = \left| \begin{array}{l} x = \sin u \Rightarrow dx = \cos u \frac{du}{d\xi}\,d\xi \\ dy = e\sin(k\xi)\,d\xi \Rightarrow y = -\frac{e}{k}\cos(k\xi) \end{array} \right|$$

$$\pi b_k = \underbrace{-\frac{e}{k}\sin u \cos(k\xi)\Big|_0^{\pi}}_{\phi} + \frac{e}{k}\int_0^{\pi}\cos u \cos(k\xi)\frac{du}{d\xi}\,d\xi = \left| \cos u \frac{du}{d\xi}\,d\xi = d(\sin u) \right|$$

$$\pi b_k = \frac{e}{k}\int_0^{\pi}\cos(k\xi)\,d(\sin u) = \frac{1}{k}\int_0^{\pi}\cos(k\xi)\,d(u - \xi) = \frac{1}{k}\int_0^{\pi}\cos(k\xi)\,du - \underbrace{\frac{1}{k}\int_0^{\pi}\cos(k\xi)\,d\xi}_{\phi,\ k\neq\phi}$$

$$\pi b_k = \frac{1}{k}\int_0^{\pi}\cos(k\xi)\,du = \frac{1}{k}\int_0^{\pi}\cos\left[k(u - e\sin u)\right]du$$

$$J_n(x) \overset{\mathrm{Def}}{=} \frac{1}{\pi}\int_0^{\pi}\cos\left[n\alpha - x\sin\alpha\right]d\alpha$$

$$b_k = \frac{J_k(ke)}{k}$$

$$\boxed{e\sin u = 2\sum_{k=1}^{\infty} J_k(ke)\,\frac{\sin(k\xi)}{k}}$$

$$t = \frac{M\sqrt{m}}{(-2E)^{3/2}}(u - e\sin u)$$

$$t = C(u - e\sin u) \Rightarrow \boxed{\xi = u - e\sin u}, \quad \xi = \frac{t}{C}$$

$$u - \xi = e\sin u = \sum_{k=1}^{\infty} b_k \sin(k\xi) \quad // \Rightarrow \begin{cases} \xi = 0 \Rightarrow (u-\xi) = 0 \\ \xi = \pi \Rightarrow (u-\xi) = 0 \end{cases}$$

$$u - \xi = \sum_{k=1}^{\infty} b_k \sin(k\xi) \; / \cdot \sin s\xi$$

$$\int_0^{\pi} (u-\xi)\sin(s\xi)\,d\xi = \sum_{k=1}^{\infty} b_k \int_0^{\pi} \sin(k\xi)\sin(s\xi)\,d\xi$$

$$\int_0^{\pi} (u-\xi)\sin(s\xi)\,d\xi = \sum_{k=1}^{\infty} b_k \frac{\pi}{2}\delta_{k,s} \Rightarrow \boxed{b_s = \frac{2}{\pi}\int_0^{\pi} (u-\xi)\sin(s\xi)\,d\xi}$$

$$\int_0^{\pi} (u-\xi)\sin(s\xi)\,d\xi = \left| \begin{array}{l} x = u - \xi \Rightarrow dx = \left(\frac{du}{d\xi} - 1\right)d\xi \\ dy = \sin(s\xi)\,d\xi \Rightarrow y = -\frac{1}{s}\cos(s\xi) \end{array} \right| =$$

$$= \frac{1}{s}\cancelto{0}{\left[(\xi - u)\cos(s\xi)\right]\Big|_0^{\pi}} + \frac{1}{s}\int_0^{\pi} \cos(s\xi)\left(\frac{du}{d\xi} - 1\right)d\xi =$$

$$= \frac{1}{s}\int_0^{\pi} \cos(s\xi)\left(\frac{du}{d\xi} - 1\right)d\xi = \frac{1}{s}\int_0^{\pi} \cos(s\xi)\,du - \cancelto{0,\ s\neq 0}{\frac{1}{s}\int_0^{\pi} \cos(s\xi)\,d\xi} = \frac{1}{s}\int_0^{\pi} \cos(s\xi)\,du =$$

$$= \frac{1}{s}\int_0^{\pi} \cos\left[s(u - e\sin u)\right]du$$

$$b_s = \frac{2}{\pi s}\int_0^{\pi} \cos\left[s(u - e\sin u)\right]du, \quad J_n(x) \overset{\text{Def}}{=} \frac{1}{\pi}\int_0^{\pi} \cos\left[x\sin\alpha - n\alpha\right]d\alpha$$

$$\boxed{b_s = 2\,\frac{1}{s}\,J_s(es)}$$

$$\boxed{u - \xi = 2\sum_{k=1}^{\infty} J_k(ek)\,\frac{\sin(k\xi)}{k}}$$

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$$\sin u = \frac{\sqrt{1-e^2}\,\sin\theta}{1+e\cos\theta} \qquad \cos u = \frac{\cos\theta + e}{1+e\cos\theta}$$

$$\cos\theta = \frac{\cos u - e}{1-e\cos u} \qquad \sin\theta = \frac{\sqrt{1-e^2}\,\sin u}{1-e\cos u}$$

$$r = a(1-e\cos u) \qquad x = r\cos\theta$$

$$t = a\sqrt{\frac{m}{2|E|}}\,(u - e\cos u) \qquad y = r\sin\theta$$

$$x = r\cos\theta = a\cancel{(1-e\cos u)}\,\frac{\cos u - e}{\cancel{1-e\cos u}} = a(\cos u - e)$$

$$y = r\sin\theta = a\cancel{(1-e\cos u)}\,\frac{\sqrt{1-e^2}\,\sin u}{\cancel{1-e\cos u}} = a\sqrt{1-e^2}\,\sin u$$

$$x = \frac{M}{-2E}(\cos u - e) \qquad \frac{x}{a} = \cos u - e \quad \dots ?$$

$$y = \sqrt{\frac{\ell^2}{-2Em}}\,\sin u \qquad \frac{y}{a} = \sqrt{1-e^2}\,\sin u$$

$$w = \sqrt{\frac{M}{a^3}} = \sqrt{\frac{M}{\frac{M^3}{(2|E|)^3}}}$$

$$\frac{y}{a} = \sqrt{1-e^2}\cdot\frac{2}{e}\sum_{k=1}^{\infty} k^{-1} J_k(ke)\sin(k\xi)$$

$$w = \sqrt{\frac{(-2E)^3}{M^2}}$$

$$\boxed{\frac{y}{a} = \frac{2}{e}\sqrt{1-e^2}\sum_{k=1}^{\infty} k^{-1} J_k(ke)\sin(kwt)} \qquad \boxed{tw = \frac{t}{c} = \xi}$$