Kepler's laws. Motion in central forces. Binary stars

Milan Milosevic, Oliwia Madej, Damir Andrasevic

1 Kepler's Laws

1.1 Celestial mechanics in simulating movement of planets

Celestial mechanics is, as it name says, the study of motion of celestial bodies like planets, stars etc. The law that explains motions of that objects in the classical universe is Newton's law of universal gravitation. So, in fact all equations of motion are based on Newton's laws.

1.2 Equation of motion

To get equation for motion of planets going around the sun we take the two body system. Masses of our bodies, supposedly a planet of our solar system and sun are marked as m_1 and m_2, and $\vec{r_1}$, $\vec{r_2}$ are their radius vectors in some fixed inertial coordinate frame. The distance of the planets is then given by the expression $\vec{r} = \vec{r_2} - \vec{r_1}$. From these facts we get the force of gravitational pull which is:

$$\vec{F} = -Gm_1m_2\frac{\vec{r}}{r^3} \tag{1}$$

where G is a gravitational constant. If we combine that equation with Newton's second law we will get an equation of motion which is

$$m_1\ddot{\vec{r_1}} = -Gm_1m_2\frac{\vec{r}}{r^3} \tag{2}$$

If we want to get a equation of relative orbit of planet or object with respect to the sun, canceling masses and subtracting we get the formula like this one

$$\ddot{\vec{r}} = -\mu\frac{\vec{r}}{r^3} \tag{3}$$

where $\mu = G(m_1 + m_2)$.
Equation (1) solves the radius vector and its second derivative. For getting a usable solution we need to express the radius vector as a function of time. There is no simple explanation how this is done but we will discuss it later.
If we want to get a geometric shape of the orbit we will have to derive the equation of the orbit. To do that we can start from equation:

$$\vec{r}\vec{e} = re\cos f \tag{4}$$

where $\vec{e}$ is a vector that points to the direction where the planet is closest to the sun in its orbit, $\vec{r}$ is a radius vector of planet and f (known as *true anomaly*) is the angle between vectors $\vec{r}$ and $\vec{e}$. Using the properties of scalar product and general properties of vector $\vec{e}$ one can get [2] general equation of conic sections (parabola, hyperbola or an ellipse) in polar coordinates:

$$r = \frac{k^2}{\mu(1 + e\cos f)} \tag{5}$$

where k is the magnitude of the angular momentum divided by the planet's mass and e is the magnitude of $\vec{e}$, known as the eccentricity of the orbit.

1.3 Orbital elements

To be able to calculate the geometry of an orbit we must first define orbital elements. So we have semi-major axis (a), eccentricity (e), inclination (i), longitude of the ascending node (Ω), argument of the perihelion (ω) and time of perihelion (τ). We can get semi-major axis if values of e and k are known:

$$a = \frac{k^2}{\mu(|1 - e^2|)} \tag{6}$$

As said in the begining, when calculating an orbit of the planets we suppose that the orbit of each planet can be approximated as a two body system with the sun. Even tough the planets interfere with each other we can calculate the orbits precise enough. But if we really want to be precise we also must take into account, the perturbations of the orbits over some time, as they accumulate.
To simulate the orbits of planet we used orbital data from ([2], table E.9) which gives the orbital elements for all planets for August 1993. Same book, table E.10, gives us the elements as polynomials, in which the variable T is the number of Julian centurys elapsed since 1900, and it is given by:

$$T = \frac{JD - 2415020}{36525} \tag{7}$$

where JD is a Julian date.

1.4 Orbit determination

The orbit is being determined with orbital elements. To compute the orbital elements we need at least three observations. Directions are usually calculated from data taken a few nights apart. With these direction we will be able to find the corresponding absolute positions, but for that we need additional constrains of the orbit. So, we assume that the object moves on the conic section lying in the plane that passes through the sun. When we get at least three radius vectors of object (one for each night) we can find the ellipse going to those three dots from our observations. The more observations we have the more accurate our result will be.

1.5 Determinating the position in the orbit

Knowing everything we know till now, we still can't find the planet at the given time as we don't know the $\vec{r}$ as a function of time what is obiouvsly a problem. We can express radius vector as

$$\vec{r} = a(\cos E - e)\vec{i} + b(\sin E)\vec{j} \tag{8}$$

Here $\vec{i}$ and $\vec{j}$ are unit vectors parallel to the major and minor axis. E is the eccentric anomaly, and e is eccentricity of the orbit; a and b are semi-major and semi-minor axis of orbit ($b = a\sqrt{1 - e^2}$. The distance between the sun and the planets, magnitude of $\vec{r}$ can be calculated from equation (8), and it is:

$$r = a(1 - e\cos E) \tag{9}$$

The value E at a moment of time is given by:

$$E - e\sin E = M \tag{10}$$

where:

$$M = \frac{2\pi(t-\tau)}{P} \tag{11}$$

For our simulation we used different ways to calculate M. From equation (7) we can calculate T and then using formulas from table E.10 at [2] (data for orbital elements) we can calculate L for each planet. Now $M = L - \tilde{\omega}$, where $\tilde{\omega}$ is longitude of perihelion.
M is mean anomaly of time which increases with constant rate and if we put it in the equation (10) we get an equation for calculating position of the planet. Equation (10 can't be solved analiticly so we must use numerical methods to solve it. This is Kepler's equation. With these we know the position of planet in the orbital plane. Simulations of orbits of planets according these equations are shown on figure (1) and (2).

2 Motion in Central Forces

2.1 Two Body Problem

We consider a system of two points of masses m_1 and m_2, in which there are forces only due to an interaction potential V. We assume that V is only a function of a position vector between m_1 and m_2. Such a system has six degree of freedom and there are six independent generalized coordinates. We suppose that these are vector coordinates of the center-of-mass $\vec{R}$, plus three components of relative vector $\vec{r} = \vec{r_1} - \vec{r_2}$. The Lagrangian of system can be written as:

$$L = T(\dot{\vec{R}}, \dot{\vec{r}}) - V(\vec{r}) \tag{12}$$

where kinetic energy T is the sum of the kinetic energy of the center-of-mass system plus kinetic energy of the motion around it, T':

$$T = \frac{1}{2}(m_1 + m_2)\dot{\vec{R}}^2 + T' \tag{13}$$

$$T' = \frac{1}{2}m_1\dot{\vec{r}}^2 + \frac{1}{2}m_2\dot{\vec{r}}'^2 \tag{14}$$

It is well known that:

$$\vec{r_1}' = -\frac{m_2}{m_1 + m_2}\vec{r} \qquad \vec{r_2}' = \frac{m_1}{m_1 + m_2}\vec{r} \tag{15}$$

Then, T' takes the form:

$$T' = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{\vec{r}}^2 \qquad (16)$$

So the Lagrangian from equation (12) becomes:

$$L = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - V(\vec{r}) \qquad (17)$$

where $M = m_1 + m_2$ is total mass, and $\mu = \frac{m_1 m_2}{m_1 + m_2}$ stands for reducted mass of the system. From equation (17) one can see that the coordinates of $\dot{\vec{R}}$ are cyclic implying that the center-of-mass is eather fixed or in uniform motion.
Now none of the equations of motion for $\vec{r}$ will contain a term where $\dot{\vec{R}}$ or $\vec{R}$ will occur, this is exactly the situation what one will have if a center of system would have been located in the center of mass with additional particle at a distance $\vec{r}$ away of mass μ.
Thus, the motion of two particles around the center of mass, which is a source of a central force, can always be reducted to equivalent problem of single body.

2.2 Equation of Motion

Now, we limit ourselves to conservative central forces for which the potential is a function only of r, $V(r)$. To solve this problem as easy as possible we can put the origin of the reference frame in the center of mass. As the potential depends only of r the problem has spherical symmetry and an angular coordinate representing that rotation should be cyclic providing another simplification to the problem. Due the spherical symmetry total angular momentum:

$$\vec{L} = \vec{r} \times \vec{p} \qquad (18)$$

is conserved. One can take the direction of $\vec{L}$ along z-axis then the motion will take place in *(x, y)* plane. The conservation of angular momentum provides three independent constant of motion, but two of them, expressing the constant direction of angular momentum are used to reduce the problem of three degree of freedom to only two.
In polar coordinates the Lagrangian is given by:

$$L = \frac{1}{2}m(\dot{r}^2 - r^2\dot{\theta}^2) - V(r) \qquad (19)$$

Now, one can put this Lagrangian in Euler-Lagrange differential equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (20)$$

Where q_i stands for generalized coordinate. We should solve equation (20) for coordinates r and θ. The angle θ is the cyclic coordinate so the one equation of motion will be:

$$\dot{p_\theta} = \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0 \qquad \Rightarrow \qquad mr^2\dot{\theta} = l = const \qquad (21)$$

where l is constant modulus of angular momentum. The conservation of angular momentum is equivalent to saying the areolar velocity is constant. That is the proof of Kepler's second law of planetary motion.
The second Lagrange equation for r coordinates reads:

$$\frac{d}{dt}(m\dot{r}) - mr^2\dot{\theta}^2 + \frac{\partial V}{\partial r} = 0 \tag{22}$$

Denoting the force by $F(r)$ and using the equation (21) equation (22) can be written as:

$$m\ddot{r} - \frac{l^2}{mr^3} = F(r) \tag{23}$$

where:

$$F(r) = m(\ddot{r} - r\dot{\theta}^2) \tag{24}$$

Recalling the conservation of the total energy:

$$E = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \tag{25}$$

it is easily to write the equation (23) as:

$$\dot{r} = \sqrt{\frac{2}{m}\left(E - T - \frac{l^2}{2mr^2}\right)} \tag{26}$$

$$dt = \frac{dr}{\sqrt{\frac{2}{m}\left(E - T - \frac{l^2}{2mr^2}\right)}} \tag{27}$$

The integration on the right side is not simple. To simplify integration one should put

$$r = \frac{M}{-2E}(1 - e\cos u) \tag{28}$$

and get:

$$t = \frac{M}{(-2E)^{\frac{3}{2}}}(u - e\sin u) \tag{29}$$

where:

$$e = \sqrt{(1 + 2E\frac{L^2}{M^2})} \tag{30}$$

is eccentricity of orbit, and for $e > 1$ $(E > 0)$ - *hyperbolic orbit*, $e = 1$ $(E = 0)$ - *parabolic orbit* and for $e < 1$ $(E < 0)$ - *elliptic orbit*.
In equations (28) and (29) u stands for the so called "mean eccentric anomaly". The connection between u and θ is given by:

$$\sin u = \frac{\sqrt{(1-e^2)}\sin\theta}{1+e\cos\theta} \qquad \cos u = \frac{\cos\theta + e}{1+e\cos\theta} \tag{31}$$

$$\cos\theta = \frac{\cos u - e}{1-e\cos u} \qquad \sin\theta = \frac{\sqrt{(1-e^2)}\sin u}{1+e\cos u} \tag{32}$$

Now the equation of orbit can be written in the form:

$$x = r\cos\theta = \frac{M}{-2E}(\cos u - e) \qquad y = r\sin\theta = \frac{L}{\sqrt{-2E}}\sin u \tag{33}$$

This equation lead to a harmonic function of time with coefficient that are standard Bessel functions [1]:

$$J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(z\sin u - nu)}du \tag{34}$$

$$\frac{x}{a} = -\frac{3}{2}e + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} k^{-1}J_{k-1}(ke)\cos k\omega t \qquad \frac{y}{a} = \sqrt{1-e^2}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} k^{-1}J_{k-1}(ke)\sin k\omega t \tag{35}$$

where

$$a = \frac{L^2/M}{1-e^2} = \frac{M}{-2E} \tag{36}$$

is samimajor axis of orbit and

$$\omega = \frac{\sqrt{-2E}^3}{M} = \sqrt{\frac{M}{a^3}}. \tag{37}$$

3 Binary Stars

Less than half of all stars are single star like sun. More than 50% belong to systems containing two or more components. Binary star systems are classified according to the data taken from observations. One of the most interesting types are eclipsing binaries. Orbital planes of its components are oriented approximately along the line of sight. In consequence one star may periodically pass in front of the other blocking the light of the eclipsed component. This kind of system can be easily recognized by regular changes of light reaching the observer. The most convenient way to analyze the observation data is to plot the light curve. This kind of function carries information about relative effective temperatures and radii of each component. It is also possible to analyze the position of orbital planes according to the position of the observer.

The aim of our project is to built a mathematical model that would fit to the hypothetical data give similar to observed light curve. For simplicity we assume inclination equal 90° (eclipses are total). Our simulation solves cases in which radii of components are either equal or different but brightness and masses stay the same. Both stars move around the center of mass on the elliptical orbits. Observations are often described as if one component remained stationary and the other orbited around it. That is why we simulate this kind of movement also. We fix our coordinate system on one star and measure velocity in different points on the orbit. Then we manage to plot the new orbit in the changed coordinate system.

3.1 Simulating the Orbits

We can apply equations (35) to simulate the orbit of binary stars. To do that the easiest approach is to use the center-of-mass system. For that system we have:

$$\mu = \frac{m}{2} \qquad M = 2m \tag{38}$$

where m is a mass of one of the stars. Now it is possible to calculate constants e, a, ω of orbit for both stars. From equations (30), (36) and (37) and than equations (35) represents a position of each star in the center-of-mass system.
The results are shown at images: (3) and (4).
When we have the equations of motion in the center-of-mass system we can change the frame of reference and put it on the first star. Then we can plot the orbit of the second star in a way the observer from the first star sees it.
The position of the stars in the center-of-mass system is given by:

$$\vec{r_1} = x_1\vec{e_1} + y_1\vec{e_2} \qquad \vec{r_2} = x_2\vec{e_1} + y_2\vec{e_2} \tag{39}$$

When $\vec{r_1}$ stands for radius vector of star, $\vec{e_1}$ and $\vec{e_2}$ are unit vector of x and y axis and (x_i, y_i) are coordinates of star. Relative vector of the second star is $\vec{e_1}$ and $\vec{e_2}$

$$\vec{r} = (x_2 - x_1)\vec{e_1} + (y_2 - y_1)\vec{e_2} \tag{40}$$

To express $\vec{r}$ in a new reference frame one have to express $\vec{e_1}$ and $\vec{e_2}$ as a function of $\vec{e_1}'$ and $\vec{e_2}'$. Consider the rotation of reference frame from $\vec{e_1}$, $\vec{e_2}$ for angle θ. Then we have

$$\vec{e_1}' = \cos\theta\vec{e_1} + \sin\theta\vec{e_2} = \sin\theta(\cot\theta\vec{e_1} + \vec{e_2}) \tag{41}$$

$$\vec{e_2}' = -\sin\theta\vec{e_1} + \cos\theta\vec{e_2} = \sin\theta(-\vec{e_1} + \cot\theta\vec{e_2}) \tag{42}$$

It is well known that:

$$\sin\theta = \frac{\tan\theta}{\sqrt{\tan^2\theta + 1}} = \frac{k}{\sqrt{k^2 + 1}} \tag{43}$$

where $k = \tan\theta$ can be determined from the equations (35) as:

$$k = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \tag{44}$$

After some mathematical transformations we finally get:

$$\vec{e_1} = \frac{1}{\sqrt{k^2 + 1}}(\vec{e_1}' - k\vec{e_2}') \qquad \vec{e_2} = \frac{1}{\sqrt{k^2 + 1}}(k\vec{e_1}' + \vec{e_2}') \tag{45}$$

and

$$\vec{r} = \left((x_2 - x_1)\frac{1}{\sqrt{k^2+1}} + (y_2 - y_1)\frac{k}{\sqrt{k^2+1}}\right)\vec{e_1} + \left(-(x_2 - x_1)\frac{k}{\sqrt{k^2+1}} + (y_2 - y_1)\frac{1}{\sqrt{k^2+1}}\right)\vec{e_2} \tag{46}$$

Where x_1, y_1, x_2 and y_2 are functions of time.
If we plot equation (46) as a function of time we can see the orbit of the second star in the reference frame fixed on the first star.
Results of this simulation are at image (5).

3.2 Simulation of Light Curve

At the end we simulate a light curve of the eclipsing binary system we analyzed before.
Time of eclipses can be determined in this way. If the observer is at coordinates (x_p, y_p) than the eclipse will be when all three points (x_p, y_p), (x_1, y_1) and (x_2, y_2) are on the straight line, or

$$(x_p - x_1)(y_2 - y_1) - (y_p - y_1)(x_2 - x_1) = 0 \tag{47}$$

The main task in simulating the light curve is the determination of the visible area during the eclipse as a function of time. The area is given by: $S = S_1 + S_2 - \triangle S$, where S_1 and S_2 are areas of first and second star and $\triangle S$ is covered area. This area, $\triangle S$, can be calculated using the rules of analytical geometry as:

$$\triangle S = 2\int_{x_1}^{x_2}\int_{0}^{\sqrt{r^2-x^2}} dxdy \tag{48}$$

After determination the visible area of star relative brightness of the system is expressed by equation:

$$\triangle m = -2.5\log\frac{f_{12}}{f_2} = -2.5\log\frac{S}{S - \triangle S} \tag{49}$$

if we assume that both stars have the same luminosity. Equation (49) represents the light curve for the simulated system of eclipsing binary stars. The light curve is shown on image (6).

References

[1] C. Misner, K. Thorne, J. Wheeler: *Gravitation*, p. 644-649

[2] H. Karttunen: *Fundamental Astronomy*

[3] B. Carroll, D. Ostlie: *An Introduction to Modern Astronomy*

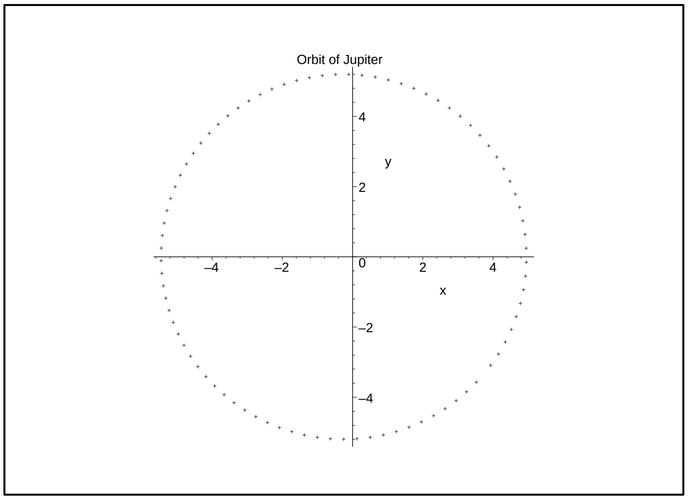

Figure 1: Orbit of Jupiter. In this simulation we used only formulas for Kepler's law and orbital elements for the planet.

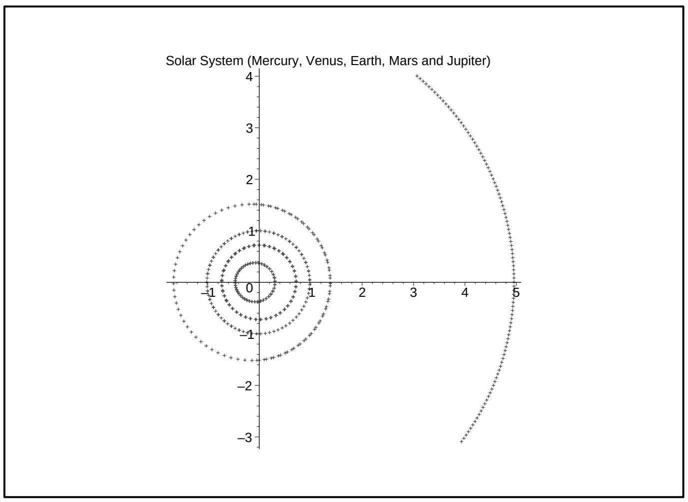

Figure 2: Simulation of orbits of first five planets in our solar system. This simulation shows the relative position of planets according to orbital datas.

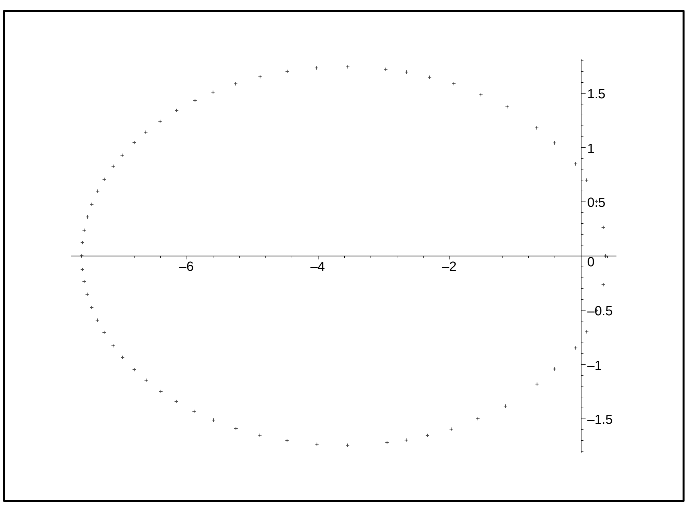

Figure 3: Simulation of orbit of one star with big eccentricity (e=*0.9*). In this case the orbits are calculated by equations (35).

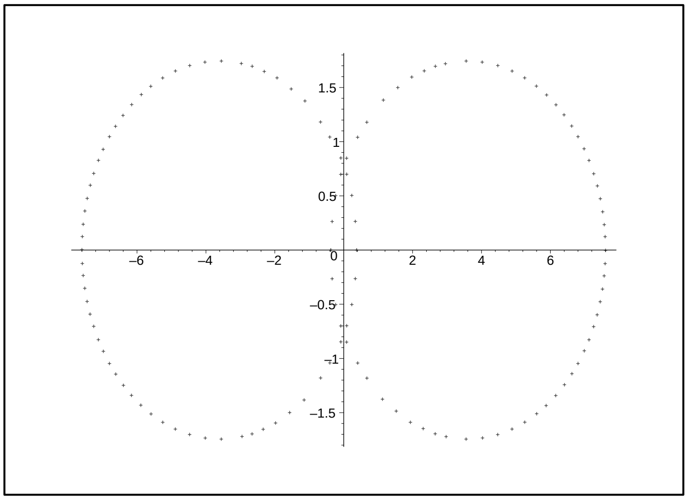

Figure 4: Orbits of stars around the center of mass in the binary system.

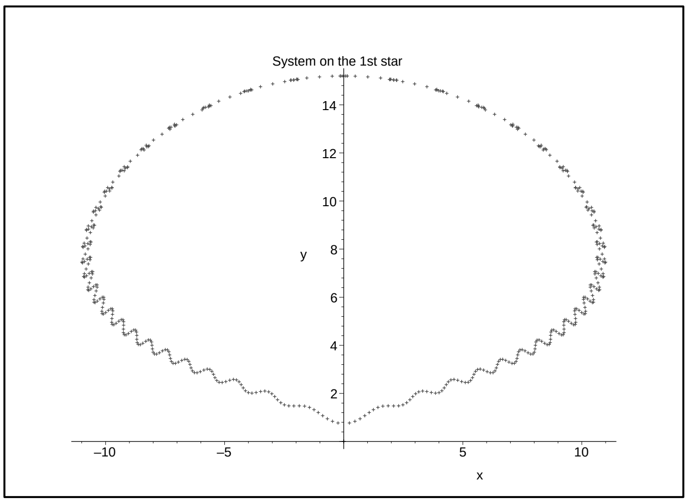

Figure 5: Simulation of orbit of the second star when the reference frame is fixed on the first star. This is the way how observer on the first star see the second star.

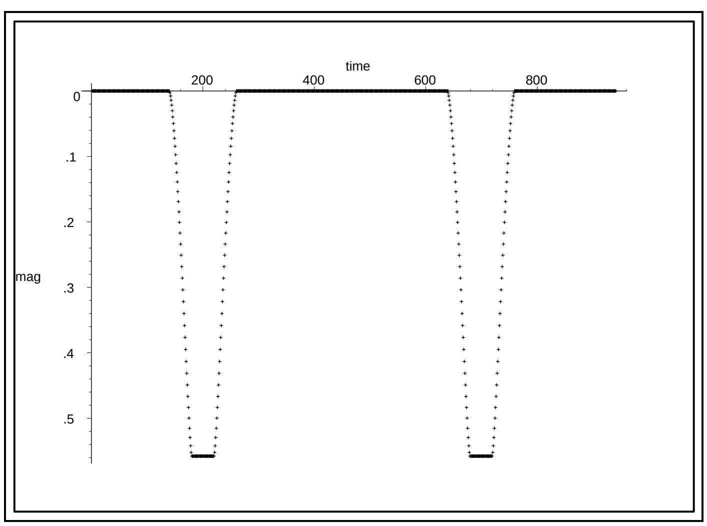

Figure 6: Light curve of an eclipsing binary system. For this simulation we assume that relative radius of stars are: $r_1 = 3$ and $r_2 = 1$. We assume that speed during the eclipse is constant, speed and time of eclipse is calculated from the simulation of orbits in center-of-mass system.